Finite dimensional semigroup quadratic algebras
with the minimal number of relations

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Abstract

A quadratic semigroup algebra is an algebra over a field given by the generators $x_1, \ldots, x_n$ and a finite set of quadratic relations each of which either has the shape $x_j x_k = 0$ or the shape $x_j x_k = x_l x_m$. We prove that a quadratic semigroup algebra given by n generators and $d \leqslant \frac{n^2+n}{4}$ relations is always infinite dimensional. This strengthens the Golod–Shafarevich estimate for the above class of algebras. Our main result however is that for every n, there is a finite dimensional quadratic semigroup algebra with n generators and δ_n relations, where δ_n is the first integer greater than $\frac{n^2+n}{4}$. That is, the above Golod–Shafarevich-type estimate for semigroup algebras is sharp.

1 Introduction

Throughout the paper, $\mathbb{K}$ is an arbitrary field and $\mathbb{N}$ is the set of positive integers. For a set X, $\mathbb{K}\langle X \rangle$ stands for the free associative algebra over $\mathbb{K}$ generated by X. For $n \in \mathbb{N}$, $\langle X \rangle_n$ denotes the set of monomials in $\mathbb{K}\langle X \rangle$ of degree n: $\langle X \rangle_n = \{x_{i_1} \ldots x_{i_n} : x_{i_j} \in X\}$.

We deal with *quadratic algebras*, that is, algebras R given as $\mathbb{K}\langle X \rangle / I$, where I is the ideal in $\mathbb{K}\langle X \rangle$ generated by a collection of homogeneous elements (called relations) of degree 2.

Algebras of this class, their growth, their Hilbert series, rationality and nil/nilpotency properties have been extensively studied (see [11, 12, 13] and references therein). One of the most challenging questions in the area (see the ICM paper [12], or [17]) is whether there exists an infinite dimensional nil algebra in this class. This is a version of the Kurosh problem with an additional constraint formulated in terms of defining relations. A version of the Kurosh problem with a restriction on the rate of growth was recently solved in [9]. As it is well-known, the original Kurosh problem was solved with the help of the Golod–Shafarevich lower estimate for the Hilbert series of an algebra. This shows that the behavior of the Hilbert series plays a crucial role in the structural theory of algebras. There is a vast literature devoted to the better understanding of the Golod–Shafarevich inequality. For instance, in [10, 3] a number of results on asymptotic (in various senses) tightness of the Golod–Shafarevich inequality is obtained. In [3] we also prove some partial non-asymptotic results results on the Anick conjecture.

The Golod–Shafarevich-type estimates have very deep connections to other areas of algebra. First, it is necessary to mention their classical applications to p-groups and class field theory [5, 6, 18]. As evidence of the connection to certain deep homological properties one can consider the fact that for quadratic algebras with n generators and d relations, the interval $\frac{n^2}{4} < d < \frac{n^2}{2}$, where the Anick conjecture on the tightness of the Golod–Shafarevich estimate is difficult to tackle, is exactly the interval, for which the generic algebra is non-Koszul (see [11]). Furthermore the notion of noncommutative complete intersection [4] can be defined in terms of the Hilbert series and refers to a class of algebras for which the Golod–Shafarevich inequality turns into equality.

On the other hand, this concept is also characterized by the nice behavior of the noncommutative Koszul (Shafarevich) complex.

In this paper, we mostly deal with quadratic semigroup algebras, which traditionally serve as a main source of examples in this area and where an interesting combinatorics of words can applied. This class was extensively studied, one can find an account of main results and methods in [8].

The Golod–Shafarevich lower estimate [5, 11] for the Hilbert series of an algebra given by quadratic relations implies [15] that R is infinite dimensional if $d \leqslant \frac{n^2}{4}$, where R is an algebra defined by n generators and d quadratic relations. Anick [1, 2] conjectured that the Golod–Shafarevich estimate is attained in the case of quadratic algebras. In particular, this conjecture, if true, implies that for all $d, n \in \mathbb{N}$ with $d > \frac{n^2}{4}$, there is a finite dimensional quadratic algebra with n generators defined by d relations. This conjecture in its full generality still remains open.

We study the same question for a subclass of the class of quadratic algebras.

Definition 1.1. A *quadratic semigroup algebra* is an algebra given by generators and relations such that each of its defining relations is either a degree 2 monomial or a difference of two degree 2 monomials.

Many partial solutions of Anick's conjecture were obtained by means of examples, which are quadratic semigroup algebras. An interesting combinatorial result dealing with semigroup relations was obtained by Wisliceny. We are moving in the same direction. Wisliceny [16] proved that the conjecture of Anick is asymptotically correct. Namely, for every $n \in \mathbb{N}$, he constructed a quadratic algebra R_n with n generators defined by d_n relations such that R_n is finite dimensional and $\lim\limits_{n \to \infty} \frac{d_n}{n^2} = \frac{1}{4}$. More specifically, $d_n = \frac{n^2 + 2n}{4}$ if n is even and $d_n = \frac{n^2 + 2n + 1}{4}$ if n is odd. The algebras R_n, constructed by Wisliceny, are quadratic semigroup algebras.

It turns out that it is not enough to consider quadratic semigroup algebras to prove Anick's conjecture. Namely, the Golod–Shafarevich estimate can be improved for this class.

Theorem 1.2. *Let R be a quadratic semigroup algebra with n generators given by $d \leqslant \frac{n^2 + n}{4}$ relations. Then R is infinite dimensional.*

However, this leaves a gap between the $\frac{n^2}{4} + \frac{n}{4}$ of Theorem 1.2 and approximately $\frac{n^2}{4} + \frac{n}{2}$ of the Wisliceny example. Surprisingly, it turns out that the estimate provided by Theorem 1.2 is tight.

Theorem 1.3. *Let $d, n \in \mathbb{N}$ be such that $d > \frac{n^2 + n}{4}$. Then there exists a finite dimensional quadratic semigroup algebra with n generators given by d relations.*

Thus the minimal possible number of defining relations of a finite dimensional quadratic semigroup algebra with n generators is precisely the first integer greater than $\frac{n^2 + n}{4}$. Theorems 1.2 and 1.3 give a solution to the natural analog of Anick's conjecture for the class of semigroup quadratic algebras. Furthermore, it roughly halves the gap between the number of relations in the original Anick's conjecture and the number of relations in Wisliceny's example, which gave the best bound known before the result of this paper.

2 Proof of Theorem 1.2

Definition 2.1. Let $g = \sum\limits_{a,b \in X} \lambda_{a,b} ab$ be a homogeneous degree 2 element of $\mathbb{K}\langle X \rangle$. Then the finite set $\{ab : \lambda_{a,b} \neq 0\} \subseteq \langle X \rangle_2$ is called *the support of g* and is denoted $\mathtt{supp}\,(g)$.

Lemma 2.2. *Let X be a non-empty set, M be a family of homogeneous degree 2 elements of $\mathbb{K}\langle X \rangle$ and $R = \mathbb{K}\langle X \rangle / I$, where I is the ideal generated by M. Assume also that there are $a, b \in X$ such that $ab, ba \notin \mathtt{supp}\,(f)$ for every $f \in M$. Then R is infinite dimensional.*

Proof. Since ab and ba are not in the support of any $f \in M$, the monomial $(ab)^n$ does not feature in the polynomial pfq for every $f \in M$ and $p, q \in \mathbb{K}\langle X \rangle$. Hence $(ab)^n \notin I$ for each $n \in \mathbb{N}$. It immediately follows that R is infinite dimensional. $\qquad\square$

Lemma 2.3. *Let X be a non-empty set, M be a family of homogeneous degree 2 elements of $\mathbb{K}\langle X \rangle$ and $R = \mathbb{K}\langle X \rangle / I$, where I is the ideal generated by M. Assume also that for each $f \in M$, the coefficients of f sum up to 0. Then R is infinite dimensional.*

Proof. Since the coefficients of each f sum up to 0, it follows that the coefficients of every element of I sum up to 0. Hence I contains no monomials and therefore R is infinite dimensional. $\qquad\square$

We are ready to prove Theorem 1.2. Denote by X the n-element set of generators of R. If there are $a, b \in X$ such that ab and ba are both not in the support of any relation, then R is infinite dimensional according to Lemma 2.2. Thus we can assume that for every $a, b \in X$ either ab or ba (or both) belongs to the union N of the supports of the relations. It follows that $|N| \geqslant \frac{n^2+n}{2}$. Since each relation is either a monomial or a difference of two monomials, their supports have at most two elements. Taking into account that we have $d \leqslant \frac{n^2+n}{4}$ relations, we are left with the only possibility that $d = \frac{n^2+n}{4}$ and each relation is a difference of two monomials. In this case, the coefficients of each relation sum up to 0. By Lemma 2.3, R is infinite dimensional. The proof of Theorem 1.2 is complete.

3   Quasierhöhungssysteme

We find the algebras from Theorem 1.3 within the following class considered by Wisliceny [16].

Definition 3.1. Let X be a finite set of generators carrying a total ordering $<$. A set $M \subset \mathbb{K}\langle X \rangle$ of homogeneous elements of degree 2 is called a *Quasierhöhungssysteme* (*QHS*) on X if

$$\bigcup_{g \in M} \mathtt{supp}\,(g) = \{ab : a, b \in X, \ a \geqslant b\} \text{ and the sets } \mathtt{supp}\,(g) \text{ are pairwise disjoint;}$$

and for every $g \in M$ one of the following three possibilities holds:

$$\text{either } g = ab \text{ with } a, b \in X, \ a \geqslant b;$$
$$\text{or } g = ab - cd \text{ with } a, b, c, d \in X, \ a \geqslant b > c \geqslant d;$$
$$\text{or } g = ab - cd \text{ with } a, b, c, d \in X, \ a > b = c > d.$$

Each QHS M generates an ideal J_M in $\mathbb{K}\langle X \rangle$ and $R_M = \mathbb{K}\langle X \rangle / J_M$ is a quadratic semigroup algebra. It is easy to see that the cardinalities of QHSs on an n-element set X range from the first integer δ_n greater than $\frac{n^2+n}{4}$ to $\frac{n^2+n}{2}$. In the subsequent sections we shall prove Theorem 1.3 by showing that there is a QHS M on an n-element X of cardinality δ_n such that the corresponding quadratic semigroup algebra R_M is finite dimensional.

This class of algebras was introduced by Wisliceny [16] and his example is from the same class. He also provided an example of a QHS M on a 6-element X for which the corresponding algebra is infinite dimensional. There is a criterion in [16] for a QHS to produce a finite dimensional algebra, but although it can be handy for treating specific examples with small number of generators with an aid of a computer, it is nearly impossible to use when X has a large number of elements. Our proof does not make use of this criterion.

Remark 3.2. Let X be a finite totally ordered set and M be a QHS on X. Let also a be the minimal element of X and b be the maximal element of X. Form the definition of a QHS it easily follows that the monomial ba belongs to M. We shall frequently deal with the set $M' = M \setminus \{ba\}$ and the ideal I_M in $\mathbb{K}\langle X \rangle$ generated by M'.

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We will also use the following notation. If I is an ideal in $\mathbb{K}\langle X \rangle$ and $f, g \in \mathbb{K}\langle X \rangle$, the equality $f = g \, (\mathrm{mod}\, I)$ means that $f - g \in I$. Furthermore, if X is totally ordered and $m \in \mathbb{N}$, we shall **always** endow the set of monomials $\langle X \rangle_m$ of degree m with the **right-to-left** lexicographical ordering. That is, $u_1 \ldots u_m < v_1 \ldots v_m$ if there is $j \in \{1, \ldots, m\}$ such that $u_j < v_j$ and $u_l = v_l$ whenever $l > j$.

Definition 3.3. Let X be a finite totally ordered set, M be a QHS on X, $a = \min X$, $b = \max X$, $M' = M \setminus \{ba\}$ and I_M be the ideal in $\mathbb{K}\langle X \rangle$ generated by M'. A monomial $u = u_1 \ldots u_m \in \langle X \rangle_m$ is called *minimal* (with respect to M) if $u \notin I_M$ and $u \leqslant v$ whenever $v \in \langle X \rangle_m$ and $u = v \, (\mathrm{mod}\, I_M)$ (in other words, u is minimal in the right-to-left lexicographical ordering among the degree m monomials belonging to the coset $u + I_M$).

We say that a minimal monomial $u \in \langle X \rangle_m$ is *tame* if there is $v \in \langle X \rangle_m$ such that $u = v \, (\mathrm{mod}\, I_M)$ and there is $j \in \{1, \ldots m\}$ such that $v_j = b$ and $v_l = u_l$ for every $l > j$.

A non-tame minimal monomial will be called *singular*.

We say that the QHS M is *regular* if there is $m \in \mathbb{N}$ such that there are no singular monomials of degree m.

Remark 3.4. It is easy to see that under the assumptions of Definition 3.3, for any $v \in \langle X \rangle_m$ such that $v \notin I_M$, there is a unique minimal monomial $u \in \langle X \rangle_m$ satisfying $v = u \, (\mathrm{mod}\, I_M)$. Furthermore, if v does not belong to the ideal J_M in $\mathbb{K}\langle X \rangle$ generated by M, then $u \notin J_M$ as well. The latter happens because $I_M \subset J_M$.

As usual, a *submonomial* of a monomial $u_1 u_2 \ldots u_m$ is a monomial of the shape $u_j u_{j+1} \ldots u_k$, where $1 \leqslant j \leqslant k \leqslant m$. From the above definition it immediately follows that a submonomial of a minimal monomial is minimal and that a minimal monomial which has a tame submonomial is tame itself. This, in turn, implies that a submonomial of a singular monomial is singular. These observations are summarized in the next lemma.

Lemma 3.5. *Let X be a finite totally ordered set and M be a QHS on X. Let also $u \in \langle X \rangle_m$ and v be a submonomial of u. Then minimality of u implies minimality of v and singularity of u implies singularity of v.*

The following lemma shows the relevance of the above definition.

Lemma 3.6. *Let X be a finite totally ordered set and M be a regular QHS on X. Then the corresponding quadratic semigroup algebra R_M is finite dimensional.*

Proof. Let $a = \min X$, $b = \max X$, $M' = M \setminus \{ab\}$, I_M and J_M be the ideals in $\mathbb{K}\langle X \rangle$, generated by M' and M respectively. Since M is regular, there is $m \in \mathbb{N}$ such that there are no singular monomials of degree m. In order to show that $R_M = \mathbb{K}\langle X \rangle / J_M$ is finite dimensional it suffices to show that R_M is $(m + 1)$-step nilpotent, or equivalently, that $\langle X \rangle_{m+1} \subset J_M$. Assume the contrary. Then (see Remark 3.4) there is a minimal monomial $u = u_1 \ldots u_m u_{m+1} \in \langle X \rangle_{m+1}$, which does not belong to J_M. By Lemma 3.5, the submonomial $\widetilde{u} = u_1 \ldots u_m$ is also minimal. Since there are no singular monomials of degree m, $\widetilde{u}$ is tame. That is, there are $j \in \{1, \ldots, m\}$ and $\widetilde{v} = v_1 \ldots v_m \in \langle X \rangle_m$ such that $\widetilde{u} = \widetilde{v} \, (\mathrm{mod}\, I_M)$, $v_j = b$ and $v_l = u_l$ whenever $j < l \leqslant m$. If $v_j u_{j+1} = b u_{j+1} \in M$, then

$$u = v_1 \ldots v_{j-1} b u_{j+1} \ldots u_{m+1} = 0 \, (\mathrm{mod}\, J_M)$$

and therefore $u \in J_M$, which is a contradiction. If $b u_{j+1} \notin M$, then since $b = \max X$, the definition of a QHS implies that there are $c, d \in X$ such that $b u_{j+1} - cd \in M'$ and $u_{j+1} > d$. Hence

$$u = v_1 \ldots v_{j-1} b u_{j+1} \ldots u_{m+1} = v_1 \ldots v_{j-1} cd u_{j+2} \ldots u_{m+1} \, (\mathrm{mod}\, I_M).$$

Since the last monomial in the above display is less than u (in the right-to-left lexicographical ordering) we have obtained a contradiction with the minimality of u. $\square$

We also need the following property of singular monomials.

Lemma 3.7. *Let X be a finite totally ordered set, $a = \min X$, $b = \max X$, M be a QHS on X, $M' = M \setminus \{ba\}$, I_M be the ideal in $\mathbb{K}\langle X \rangle$ generated by M' and $u = u_1 \ldots u_m \in \langle X \rangle_m$ be a singular monomial. Assume also that $1 \leqslant k \leqslant m$ and $v = v_1 \ldots v_k \in \langle X \rangle_k$ is such that $v = u_1 \ldots u_k \pmod{I_M}$. Then there exists $w = w_1 \ldots w_m \in \langle X \rangle_m$ such that $w_m \geqslant v_k$ and $u = w \pmod{I_M}$.*

Proof. We shall use induction by $m - k$. If $m - k = 0$, then $m = k$ and $w = v$ satisfies all desired conditions. Thus we have our basis of induction. Assume now that $n \in \mathbb{N}$ and that the statement of the lemma holds whenever $m - k < n$. We have to prove it the case $m - k = n$.

First, consider the case $u_{k+1} \geqslant v_k$. Since $v_1 \ldots v_k u_{k+1} = u_1 \ldots u_k u_{k+1} \pmod{I_M}$ and the difference between the degrees of u and $v_1 \ldots v_k u_{k+1}$ is $n - 1$, the induction hypothesis provides us with $w = w_1 \ldots w_m \in \langle X \rangle_m$ such that $w_m \geqslant u_{k+1}$ and $u = w \pmod{I_M}$. Since $u_{k+1} \geqslant v_k$, we have $w_m \geqslant v_k$, as required.

It remains to consider the case $u_{k+1} < v_k$. Observe that $v_k \neq b$. Indeed, otherwise the equality

$$v_1 \ldots v_{k-1} v_k u_{k+1} \ldots u_m = u \pmod{I_M} \tag{3.1}$$

implies that u is tame. Since $u_{k+1} < v_k$, the degree 2 monomial $v_k u_{k+1}$ features in exactly one of the supports of the elements of M (see the definition of a QHS). Since $v_k \neq b$, $v_k u_{k+1} \neq ba$ and therefore there is exactly one $g \in M'$ such that $v_k u_{k+1} \in \mathtt{supp}\,(g)$. There are three possibilities: either $g = v_k u_{k+1}$ or $g = v_k u_{k+1} - cd$ with $d < u_{k+1}$ or $g = cd - v_k u_{k+1}$ with $d \geqslant v_k$. If $g = v_k u_{k+1}$, we have $v_k u_{k+1} \in I_M$ and therefore (3.1) implies that $u \in I_M$, which contradicts the minimality of u. Hence $g = v_k u_{k+1} - cd$ with $d < u_{k+1}$ or $g = cd - v_k u_{k+1}$ with $d \geqslant v_k$. In either case $v_k u_{k+1} = cd \pmod{I_M}$ and (3.1) implies that

$$u = v_1 \ldots v_{k-1} cd u_{k+2} \ldots u_m \pmod{I_M}. \tag{3.2}$$

In the case $d < u_{k+1}$ the monomial in the right-hand-side of (3.2) is less than u, which contradicts the minimality of u. It remains to consider the case $d \geqslant v_k$. Since the difference between the degrees of u and $v_1 \ldots v_{k-1} cd$ is $n - 1$, by the induction hypothesis, there is $w = w_1 \ldots w_m \in \langle X \rangle_m$ such that $w_m \geqslant d$ and $u = w \pmod{I_M}$. Since $d \geqslant v_k$, we have $w_m \geqslant v_k$, as required. $\square$

We also need a slight improvement of Lemma 3.7 in a particular case.

Definition 3.8. Let X be a finite totally ordered set and M be a QHS on X. We say that $d \in X$ is *pure* if $a \geqslant b > c \geqslant d$ whenever $ab - cd \in M$.

Lemma 3.9. *Let X be a finite totally ordered set, $a = \min X$, $b = \max X$, M be a QHS on X, $M' = M \setminus \{ba\}$, I_M be the ideal in $\mathbb{K}\langle X \rangle$ generated by M' and $u = u_1 \ldots u_m \in \langle X \rangle_m$ be a singular monomial such that u_m is pure. Assume also that $1 \leqslant k < m$ and $v = v_1 \ldots v_k \in \langle X \rangle_k$ is such that $v = u_1 \ldots u_k \pmod{I_M}$. Then there exists $w = w_1 \ldots w_m \in \langle X \rangle_m$ such that $w_m > v_k$ and $u = w \pmod{I_M}$.*

Proof. If $u_m > v_k$, we can just take $w = u$. Thus we can assume that $v_k \geqslant u_m$. By Lemma 3.5, $u_1 \ldots u_{m-1}$ is singular and therefore Lemma 3.7 provides $y = y_1 \ldots y_{m-1} \in \langle X \rangle_{m-1}$ such that $y = u_1 \ldots u_{m-1} \pmod{I_M}$ and $y_{m-1} \geqslant v_k$. Clearly,

$$u = y_1 \ldots y_{m-2} y_{m-1} u_m \pmod{I_M}. \tag{3.3}$$

Since $y_{m-1} \geqslant v_k \geqslant u_m$, the monomial $y_{m-1} u_m$ features in one of the elements of M. Next, $y_{m-1} \neq b$ (otherwise (3.3) implies that u is tame). Hence $y_{m-1} u_m \neq ba$ and therefore $y_{m-1} u_m$ belongs to the support of some $g \in M'$. Taking into account that u_m is pure, we have three possibilities: either $g = y_{m-1} u_m$ or $g = y_{m-1} u_m - cd$ with $d < u_m$ or $g = cd - y_{m-1} u_m$ with $d > y_{m-1}$.

If $g = y_{m-1}u_m$, $y_{m-1}u_m \in I_M$ and therefore (3.3) implies that $u \in I_M$, which contradicts the minimality of u. Hence $g = y_{m-1}u_m - cd$ with $d < u_m$ or $g = cd - y_{m-1}u_m$ with $d > y_{m-1}$. In either case $y_{m-1}u_m = cd \,(\mathrm{mod}\, I_M)$ and (3.3) gives

$$u = y_1 \dots y_{m-2} cd \,(\mathrm{mod}\, I_M). \tag{3.4}$$

In the case $d < u_{k+1}$ the monomial in the right-hand-side of (3.4) is less than u, which contradicts the minimality of u. It remains to consider the case $d > y_{m-1} \geqslant v_k$. In this case $w = y_1 \dots y_{m-2} cd$ satisfies all desired conditions. $\quad\square$

Corollary 3.10. *Let X be an n-element totally ordered set, $a = \min X$, $b = \max X$, M be a QHS on X, $M' = M \setminus \{ba\}$, I_M be the ideal in $\mathbb{K}\langle X \rangle$ generated by M' and $u = u_1 \dots u_m \in \langle X \rangle_m$ be a singular monomial. Then the set $\{j : u_j$ is pure$\}$ has at most $n - 1$ elements.*

Proof. Assume the contrary. Then we can choose $j_1, \dots, j_n \in \mathbb{N}$ such that $1 \leqslant j_1 < \dots < j_n \leqslant n$ and each u_{j_k} is pure. Let $a_1 = u_{j_1}$ and take $y^{(1)} = u_1 \dots u_{j_1-1}$. Clearly $u_1 \dots u_{j_1} = y^{(1)}a_1$ and therefore $u_1 \dots u_{j_1} = y^{(1)}a_1 \,(\mathrm{mod}\, I_M)$. By Lemma 3.9, there are $a_2 > a_1$ and a monomial $y^{(2)} \in \langle X \rangle_{j_2-1}$ such that $u_1 \dots u_{j_2} = y^{(2)}a_2 \,(\mathrm{mod}\, I_M)$. Applying Lemma 3.9 again, we see that there are $a_3 > a_2$ and a monomial $y^{(3)} \in \langle X \rangle_{j_3-1}$ such that $u_1 \dots u_{j_3} = y^{(3)}a_3 \,(\mathrm{mod}\, I_M)$. Proceeding this way we obtain $a_1, \dots, a_n \in X$ and the monomials $y^{(k)} \in \langle X \rangle_{j_k-1}$ such that $u_1 \dots u_{j_k} = y^{(k)}a_k \,(\mathrm{mod}\, I_M)$ for $1 \leqslant k \leqslant n$ and, most importantly, $a_1 < \dots < a_n$. Since X has n elements, the latter can only happen if $a_n = b$. Then the equality $u_1 \dots u_{j_n} = y^{(n)}a_n \,(\mathrm{mod}\, I_M)$ implies that $u = y^{(n)}bu_{j_n+1} \dots u_m \,(\mathrm{mod}\, I_M)$, which means that u is tame. This contradiction completes the proof. $\quad\square$

4 The extension construction for a Quasierhöhungssysteme

Throughout this section $n \geqslant 5$ is an integer, $X = \{x_1, \dots, x_n\}$ is an n-element set with the total ordering $x_1 < x_2 < \dots < x_n$ and the $(n-4)$-element set $X_0 = X \setminus \{x_1, x_2, x_{n-1}, x_n\} = \{x_3, \dots, x_{n-2}\}$ carries the total ordering inherited from X. Let also M_0 be a QHS on X_0. By Remark 3.2, $x_{n-2}x_3 \in M_0$. As usual, $M'_0 = M_0 \setminus \{x_{n-2}x_3\}$.

We consider $M, M' \subset \mathbb{K}\langle X \rangle$ defined in the following way:

$$M' = M'_0 \cup \{x_n x_j - x_j x_1 : 2 \leqslant j \leqslant n-2\} \cup \{x_{n-1}x_{j+1} - x_j x_2 : 2 \leqslant j \leqslant n-3\}$$
$$\cup \{x_n x_n - x_{n-1}x_1, \ x_n x_{n-1} - x_{n-2}x_2, \ x_{n-1}x_{n-1} - x_{n-2}x_3, \ x_{n-1}x_2 - x_1 x_1\}$$

and

$$M = M' \cup \{x_n x_1\}.$$

It is straightforward to verify that M is a QHS on X.

Symbols J and I stand for the ideals in $\mathbb{K}\langle X \rangle$ generated by M and M' respectively. Similarly, J_0 and I_0 are the ideals in $\mathbb{K}\langle X_0 \rangle$ generated by M_0 and M'_0 respectively.

Lemma 4.1. *Let $m \in \mathbb{N}$ and $u = u_1 u_2 \dots u_m \in \langle X \rangle_m$ be a singular (with respect to M) monomial. Then the set $\{j : u_j \in \{x_2, x_n, x_{n-1}\}\}$ has at most $n - 1$ elements.*

Proof. From the way we defined M it easily follows that x_n, x_{n-1} and x_2 are pure. It remains to apply Corollary 3.10. $\quad\square$

Lemma 4.2. *Let k be a non-negative integer and $a \in \{x_2, \dots, x_{n-2}\}$. Then $x_n^k a = ax_1^k \,(\mathrm{mod}\, I)$.*

Proof. From the definition of M it follows that $x_n a - ax_1 \in M' \subset I$. That is, $x_n a = ax_1 \,(\mathrm{mod}\, I)$. The required inequality now follows via an obvious inductive argument. $\quad\square$

Lemma 4.3. *Let* $q = \frac{5n-9}{2}$ *if* n *is odd and* $q = \frac{5n-8}{2}$ *if* n *is even. Then there exists a monomial* $u = u_1 \ldots u_q \in \langle X \rangle_q$ *such that* $u_q = x_n$ *and* $u = x_1^q \pmod I$.

Proof. Since $x_{n-1}x_2 - x_1x_1 \in M' \subset I$, we see that for every $k \geqslant 2$, $x_1^k = x_{n-1}x_2x_1^{k-2} \pmod I$. By Lemma 4.2, $x_2x_1^{k-2} = x_n^{k-2}x_2 \pmod I$. Hence

$$x_1^k = x_{n-1}x_n^{k-2}x_2 \pmod I \text{ for every } k \geqslant 2. \tag{4.1}$$

Since $x_nx_n - x_{n-1}x_1 \in M'$ and $x_nx_{n-1} - x_{n-2}x_2 \in M'$, we get $x_n^3 = x_nx_{n-1}x_1 = x_{n-2}x_2x_1 \pmod I$. Hence $x_n^k = x_n^{k-3}x_{n-2}x_2x_1 \pmod I$. By Lemma 4.2, $x_n^{k-3}x_{n-2} = x_{n-2}x_1^{k-3} \pmod I$. Combining the last two equalities, we obtain

$$x_n^k = x_{n-2}x_1^{k-3}x_2x_1 \pmod I \text{ for every } k \geqslant 3. \tag{4.2}$$

For $k \geqslant 5$, we can apply (4.2) to x_n^{k-2} in (4.1), which gives

$$x_1^k = x_{n-1}x_{n-2}x_1^{k-5}x_2x_1x_2 \pmod I \text{ for every } k \geqslant 5. \tag{4.3}$$

Since $x_nx_2 - x_2x_1 \in M'$ and $x_{n-1}x_3 - x_2x_2 \in M'$, we have

$$x_2x_1x_2 = x_nx_2x_2 = x_nx_{n-1}x_3 \pmod I. \tag{4.4}$$

Next, for $2 \leqslant j \leqslant n-4$, we have $x_{n-1}x_{j+1} - x_jx_2 \in M'$, $x_nx_{j+1} - x_{j+1}x_1 \in M'$ and $x_{n-1}x_{j+2} - x_{j+1}x_2 \in M'$. Hence,

$$x_jx_2x_1x_2 = x_{n-1}x_{j+1}x_1x_2 = x_{n-1}x_nx_{j+1}x_2 = x_{n-1}x_nx_{n-1}x_{j+2} \pmod I \text{ for } 2 \leqslant j \leqslant n-4. \tag{4.5}$$

Finally, since $x_{n-1}x_{n-2} - x_{n-3}x_2 \in M'$, $x_nx_{n-2} - x_{n-2}x_1 \in M'$, $x_nx_{n-1} - x_{n-2}x_2 \in M'$ and $x_nx_n - x_{n-1}x_1 \in M'$, we get

$$\begin{aligned} x_{n-3}x_2x_1x_2x_1 &= x_{n-1}x_{n-2}x_1x_2x_1 = x_{n-1}x_nx_{n-2}x_2x_1 = \\ &= x_{n-1}x_nx_nx_{n-1}x_1 = x_{n-1}x_nx_nx_nx_n \pmod I. \end{aligned} \tag{4.6}$$

Case 1: n is odd. In this case $n = 2k+3$ and $q = 5k+3$ for some $k \in \mathbb{N}$. Using (4.3) consecutively k times, we see that

$$x_1^q = (x_{n-1}x_{n-2})^k x_1x_1(x_2x_1x_2)^k x_1 \pmod I.$$

Since $x_1x_1 = x_{n-1}x_2 \pmod I$, we obtain

$$x_1^q = (x_{n-1}x_{n-2})^k x_{n-1}x_2(x_2x_1x_2)^k x_1 \pmod I.$$

Applying (4.5) consecutively $k-1$ times, we have

$$x_1^q = (x_{n-1}x_{n-2})^k x_{n-1}(x_{n-1}x_nx_{n-1})^{k-1} x_{n-3}x_2x_1x_2x_1 \pmod I.$$

According to (4.6),

$$x_1^q = (x_{n-1}x_{n-2})^k x_{n-1}(x_{n-1}x_nx_{n-1})^{k-1} x_{n-1}x_nx_nx_nx_n \pmod I,$$

which completes the proof in the case of odd n.

Case 2: n is even. In this case $n = 2k+2$ and $q = 5k+1$ for some $k \geqslant 2$. Using (4.3) consecutively k times, we see that

$$x_1^q = (x_{n-1}x_{n-2})^k (x_2x_1x_2)^k x_1 \pmod I.$$

By (4.4),
$$x_1^q = (x_{n-1}x_{n-2})^k x_n x_{n-1} x_3 (x_2 x_1 x_2)^{k-1} x_1 \pmod I.$$

Applying (4.5) consecutively $k-2$ times, we have
$$x_1^q = (x_{n-1}x_{n-2})^k x_n x_{n-1} (x_{n-1}x_n x_{n-1})^{k-2} x_{n-3} x_2 x_1 x_2 x_1 \pmod I.$$

According to (4.6),
$$x_1^q = (x_{n-1}x_{n-2})^k x_n x_{n-1} (x_{n-1}x_n x_{n-1})^{k-2} x_{n-1} x_n x_n x_n x_n \pmod I,$$

which completes the proof in the case of even n. $\qquad\square$

Lemma 4.4. *Let k be a non-negative integer. If $ab \in M_0'$, then $ax_1^k b \in I$. If $ab - cd \in M_0'$, then $ax_1^k b = cx_1^k d \pmod I$.*

Proof. If $ab \in M_0'$, we have $ab \in I_0 \subset I$ and therefore Lemma 4.2 implies that $ax_1^k b = x_n^k ab \in I$.
  If $ab - cd \in M_0'$, we can use Lemma 4.2 to see that $ax_1^k b = x_n^k ab = x_n^k cd = cx_1^k d \pmod I$. $\qquad\square$

Since I_0 is the ideal in $\mathbb{K}\langle X_0 \rangle$ generated by M_0', the above lemma immediately implies the following result.

Corollary 4.5. *Let $u, v \in \langle X_0 \rangle_m$ and $u = v \pmod{I_0}$. Then $\widetilde{u} = \widetilde{v} \pmod I$ for every non-negative integers $k_1, \ldots, k_{m-1}$, where $u = u_1 \ldots u_m$, $v = v_1 \ldots, v_m$, $\widetilde{u} = u_1 x_1^{k_1} u_2 x_1^{k_2} \ldots x_1^{k_{m-1}} u_m$ and $\widetilde{v} = v_1 x_1^{k_1} v_2 x_1^{k_2} \ldots x_1^{k_{m-1}} v_m$.*

Lemma 4.6. *Let $m \in \mathbb{N}$, $m \geqslant 3$ and $u_1, \ldots, u_m \in \langle X_0 \rangle_m$ be such that the monomial $u_1 \ldots u_{m-2}$ is M_0-tame. Let also $k_1, \ldots, k_{m-1}$ be non-negative integers and $w = u_1 x_1^{k_1} u_2 x_1^{k_2} \ldots x_1^{k_{m-1}} u_m$. Then w is not M-singular.*

Proof. Since $u_1 \ldots u_{m-2}$ is M_0-tame, there are $v_1 \ldots v_{m-2} \in \langle X_0 \rangle_{m-2}$ and $j \in \{1, \ldots, m-2\}$ such that
$$u_1 \ldots u_{m-2} = v_1 \ldots v_{m-2} \pmod{I_0}, \quad v_j = x_{n-2} \text{ and } u_l = v_l \text{ for } j < l \leqslant m-2. \tag{4.7}$$

According to (4.7) and Corollary 4.5,
$$w = v_1 x_1^{k_1} \ldots v_{j-1} x_1^{k_{j-1}} x_{n-2} x_1^{k_j} u_{j+1} x_1^{k_{j+1}} \ldots u_{m-1} x_1^{k_{m-1}} u_m \pmod I.$$

Applying Lemma 4.2 to $x_{n-2} x_1^{k_j}$ and using the above display, we obtain
$$w = y x_{n-2} u_{j+1} x_1^{k_{j+1}} \ldots u_{m-1} x_1^{k_{m-1}} u_m \pmod I, \text{ where } y = v_1 x_1^{k_1} \ldots v_{j-1} x_1^{k_{j-1}} x_n^{k_j}. \tag{4.8}$$

Since $x_{n-2} = \max X_0$, the monomial $x_{n-2} u_{j+1}$ features in exactly one $g \in M_0$. If $u_{j+1} > x_3$, $g \in M_0'$ and there are two possibilities: either $g = x_{n-2} u_{j+1}$ or $g = x_{n-2} u_{j+1} - cd$ with $d < u_{j+1}$. In the first case (4.8) implies that $w \in I$ and therefore w is non-minimal and therefore non-singular. In the second case (4.8) gives
$$w = y cd x_1^{k_{j+1}} \ldots u_{m-1} x_1^{k_{m-1}} u_m \pmod I.$$

The monomial in the right-hand side of the above display is less than w and again we see that w is non-minimal and therefore non-singular.
  It remains to consider the case $u_{j+1} = x_3$. Since $x_{n-1}x_{n-1} - x_{n-2}x_3 \in M'$, we have $x_{n-2}x_3 = x_{n-1}x_{n-1} \pmod I$. Thus in this case (4.8) implies that
$$w = y x_{n-1} x_{n-1} x_1^{k_{j+1}} \ldots u_{m-1} x_1^{k_{m-1}} u_m \pmod I. \tag{4.9}$$

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Again, we have two possibilities: either $k_{j+1} > 0$ or $k_{j+1} = 0$. First, assume that $k_{j+1} = 0$. In this case (4.9) reads

$$w = yx_{n-1}x_{n-1}u_{j+2}x_1^{k_{j+2}} \ldots u_{m-1}x_1^{k_{m-1}}u_m \,(\mathrm{mod}\, I).$$

Since $u_{j+2} \in X_0$, from the definition of M it follows that there is $g \in M'$ such that $g = x_{n-1}u_{j+2} - cd$ with $d < u_{j+1}$. Then $x_{n-1}u_{j+2} = cd \,(\mathrm{mod}\, I)$ and according to the above display

$$w = yx_{n-1}cdx_1^{k_{j+2}} \ldots u_{m-1}x_1^{k_{m-1}}u_m \,(\mathrm{mod}\, I).$$

The monomial in the right-hand side of the above display is less than w and therefore w is non-minimal and hence non-singular. Finally, assume that $k_{j+1} > 0$. Since $x_n x_n - x_{n-1}x_1 \in M'$, we have $x_{n-1}x_1 = x_n x_n \,(\mathrm{mod}\, I)$ and (4.9) implies that

$$w = yx_{n-1}x_n x_n x_1^{k_{j+1}-1}u_{j+3}x_1^{k_{j+3}} \ldots u_{m-1}x_1^{k_{m-1}}u_m \,(\mathrm{mod}\, I).$$

The above display shows that w is tame if it is minimal. Thus in any case, w is non-singular. $\square$

The main result of this section is the following theorem.

Theorem 4.7. *If M_0 is regular, then M is also regular.*

Proof. Assume that M_0 is regular. Then there is $m \in \mathbb{N}$ such that there are no M_0-singular monomials in $\langle X_0 \rangle_{m-2}$. Let $q = \frac{5n-9}{2}$ if n is odd and $q = \frac{5n-8}{2}$ if n is even and set $r = nmq$.

It suffices to show that there are no M-singular monomials in $\langle X \rangle_r$. Assume that there exists an M-singular $u \in \langle X \rangle_r$. By Lemma 4.1, the set $\{j : u_j \in \{x_2, x_n, x_{n-1}\}\}$ has at most $n-1$ elements. It follows that there is a submonomial v of u of degree mq such that v does not contain x_2, x_{n-1} or x_n. By Lemma 3.5, v is singular. If at least m entries of v belong to X_0, Lemma 4.6 implies that v is non-singular and we arrive to a contradiction. Thus the degree mq monomial v contains at most $m-1$ entries from X_0. Since all other entries of v are x_1, it follows that x_1^q is a submonomial of v. Hence x_1^q is singular, which contradicts Lemma 4.3. $\square$

It is straightforward to see that the cardinalities of M and M_0 are related by the equality $|M| = |M_0| + 2n - 3$. Hence Theorem 4.7 immediately implies the following result.

Corollary 4.8. *Suppose that $n \geqslant 5$ and that there is a k-element regular QHS on an $(n-4)$-element set. Then there is a $(k + 2n - 3)$-element regular QHS on an n-element set.*

Applying Corollary 4.8 several times, we, after an exercise on summing up an arithmetic series, obtain the following result.

Proposition 4.9. *Let $m, j \in \mathbb{N}$ and there is a k-element regular QHS on an m-element set. Then there is a $(k + 2jm + 4j^2 + j)$-element regular QHS on an $(m + 4j)$-element set.*

5 Proof of Theorem 1.3

We start with the following 4 specific examples of QHS. For $1 \leqslant m \leqslant 4$, we set $X_m = \{x_1, \ldots, x_m\}$ ordered by $x_1 < x_2 < x_3 < x_4$. Now we define a QHS M_m on the m-element totally ordered set X_m:

$M_1 = \{x_1 x_1\}, \quad M_2 = \{x_2 x_2 - x_1 x_1, x_2 x_1\}, \quad M_3 = \{x_3 x_3 - x_2 x_1, x_3 x_2 - x_1 x_1, x_3 x_1, x_2 x_2\},$
$M_4 = \{x_4 x_4 - x_3 x_1, x_4 x_3 - x_2 x_1, x_4 x_2 - x_1 x_1, x_3 x_3 - x_2 x_2, x_3 x_2, x_4 x_1\}.$

Lemma 5.1. *For each $m \in \{1, 2, 3, 4\}$, the QHS M_m is regular.*

Proof. It is straightforward to see that for each M_m, each element of X_m is pure. By Corollary 3.10 an M_m-singular monomial can not have the degree more than $m-1$. Hence there are no M_m-singular monomials of degree m. By definition, each M_m is regular. $\square$

Let j be an arbitrary non-negative integer. By Lemma 5.1, there is a regular 1-element QHS on a 1-element set. By Proposition 4.9, there is a $(4j^2+3j+1)$-element regular QHS on a $(4j+1)$-element set. By Lemma 5.1, there is a regular 2-element QHS on a 2-element set. By Proposition 4.9, there is a $(4j^2+5j+2)$-element regular QHS on a $(4j+2)$-element set. By Lemma 5.1, there is a regular 4-element QHS on a 3-element set. By Proposition 4.9, there is a $(4j^2+7j+4)$-element regular QHS on a $(4j+3)$-element set. Finally, by Lemma 5.1, there is a regular 6-element QHS on a 4-element set. By Proposition 4.9, there is a $(4j^2+9j+6)$-element regular QHS on a $(4j+4)$-element set.

For $n \in \mathbb{N}$, we define $\delta_n = 4j^2+3j+1$ if $n = 4j+1$, $\delta_n = 4j^2+5j+2$ if $n = 4j+2$, $\delta_n = 4j^2+7j+4$ if $n = 4j+3$ and $\delta_n = 4j^2+9j+6$ if $n = 4j+4$. The above observations mean that for every $n \in \mathbb{N}$, there is a δ_n-element regular QHS on an n-element set. According to Lemma 3.6, this means that for every $n \in \mathbb{N}$, there is a quadratic semigroup algebra R_n with n generators given by δ_n relations such that R_n is finite dimensional. On the other hand, a routine check shows that δ_n is precisely the smallest integer greater than $\frac{n^2+n}{4}$. This completes the proof of Theorem 1.3.

We conclude with the following remarks.

Remark 5.2. We have just seen that for $n \leqslant 4$, there is a δ_n-element QHS on an n-element set X for which every $c \in X$ is pure. It turns out that $n = 4$ is the last n for which this phenomenon occurs. More precisely, one can easily see that a QHS on an n-element set X, for which every $c \in X$ is pure, must contain at least $\frac{n^2+2n}{4}$ elements.

Remark 5.3. Vershik [14] conjectured that for every $n \geqslant 3$, there is a finite dimensional quadratic algebra with n generators given by $\frac{n^2-n}{2}$ relations. This conjecture is proved in [7] (see also [3] for the proof in the case $3 \leqslant n \leqslant 7$). Note that for $n \geqslant 4$, this statement could also be obtained as a consequence of Theorem 1.3.

Remark 5.4. It would be interesting to get a tight enough estimate of the order of nilpotency of quadratic semigroup algebras which we construct while proving Theorem 1.3. An estimate, that follows from the proof is way higher than the actual value for small n.

Remark 5.5. Let R be a quadratic algebra with n generators given by $d \leqslant n^2$ relations. For $k \in \mathbb{N}$, R_k stands for the degree k homogeneous component of R. By the Golod–Shafarevich estimate $\dim R_3 \geqslant \min\{0, n^3 - 2dn\}$. Anick [1, 2] proved that this estimate for $\dim R_3$ is actually attained for every n and d. An analysis of his proof shows that it is also attained if we restrict ourselves to semigroup algebras.

On the other hand (see [7]), there is an R with $n = d = 3$ for which $R_5 = \{0\}$, while Theorem 1.2 implies that $\dim R_5 > 0$ if R is a semigroup algebra with $n = d = 3$. So, for every pair (n, d), the minimal dimensions of the third component for general quadratic algebras and for semigroup quadratic algebras coincide, while the same statement fails for the dimension of the fifth component. The situation with the fourth component remains unclear. Its clarification is a part of the following much more general question.

Question 5.6. *What is the minimal dimension of R_k, where R is a semigroup quadratic algebra with n generators given by $d \leqslant n^2$ relations?*

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